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Diploma Programme
Programme du diplôme
Programa del Diploma

Mathematics: analysis and approaches
Higher level
Paper 3

Thursday 12 May 2022 (morning)

1 hour

Instructions to candidates

- Do not open this examination paper until instructed to do so.
- A graphic display calculator is required for this paper.
- Answer all the questions in the answer booklet provided.
- Unless otherwise stated in the question, all numerical answers should be given exactly or correct to three significant figures.
- A clean copy of the **mathematics: analysis and approaches formula booklet** is required for this paper.
- The maximum mark for this examination paper is **[55 marks]**.

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Answer **all** questions in the answer booklet provided. Please start each question on a new page. Full marks are not necessarily awarded for a correct answer with no working. Answers must be supported by working and/or explanations. Solutions found from a graphic display calculator should be supported by suitable working. For example, if graphs are used to find a solution, you should sketch these as part of your answer. Where an answer is incorrect, some marks may be given for a correct method, provided this is shown by written working. You are therefore advised to show all working.

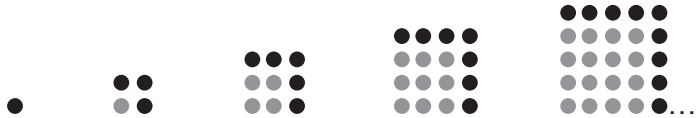
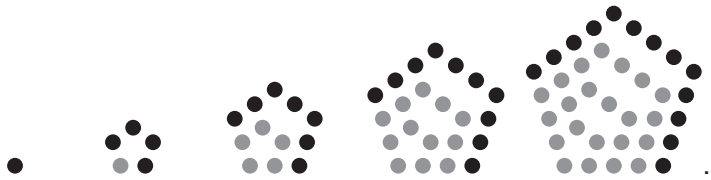
1. [Maximum marks: 27]

This question asks you to explore some properties of polygonal numbers and to determine and prove interesting results involving these numbers.

A polygonal number is an integer which can be represented as a series of dots arranged in the shape of a regular polygon. Triangular numbers, square numbers and pentagonal numbers are examples of polygonal numbers.

For example, a triangular number is a number that can be arranged in the shape of an equilateral triangle. The first five triangular numbers are $1, 3, 6, 10$ and 15.

The following table illustrates the first five triangular, square and pentagonal numbers respectively. In each case the first polygonal number is one represented by a single dot.

Type of polygonal number	**Geometric representation**	**Values**
Triangular numbers		1, 3, 6, 10, 15, ...
Square numbers		1, 4, 9, 16, 25, ...
Pentagonal numbers		1, 5, 12, 22, 35, ...

For an r-sided regular polygon, where $r \in \mathbb{Z}^+$, $r \geq 3$, the nth polygonal number $P_r(n)$ is given by

$$P_r(n) = \frac{(r-2)n^2 - (r-4)n}{2}, \text{ where } n \in \mathbb{Z}^+.$$

(This question continues on the following page)

(Question 1 continued)

Hence, for square numbers, $P_4(n) = \dfrac{(4-2)n^2 - (4-4)n}{2} = n^2$.

(a) (i) For triangular numbers, verify that $P_3(n) = \dfrac{n(n+1)}{2}$. [2]

(ii) The number 351 is a triangular number. Determine which one it is. [2]

(b) (i) Show that $P_3(n) + P_3(n+1) \equiv (n+1)^2$. [2]

(ii) State, in words, what the identity given in part (b)(i) shows for two consecutive triangular numbers. [1]

(iii) For $n = 4$, sketch a diagram clearly showing your answer to part (b)(ii). [1]

(c) Show that $8P_3(n) + 1$ is the square of an odd number for all $n \in \mathbb{Z}^+$. [3]

The nth pentagonal number can be represented by the arithmetic series

$$P_5(n) = 1 + 4 + 7 + ... + (3n - 2).$$

(d) Hence show that $P_5(n) = \dfrac{n(3n-1)}{2}$ for $n \in \mathbb{Z}^+$. [3]

(e) By using a suitable table of values or otherwise, determine the smallest positive integer, greater than 1, that is both a triangular number and a pentagonal number. [5]

A polygonal number, $P_r(n)$, can be represented by the series

$$\sum_{m=1}^{n}\big(1+(m-1)(r-2)\big) \text{ where } r \in \mathbb{Z}^+,\ r \geq 3.$$

(f) Use mathematical induction to prove that $P_r(n) = \dfrac{(r-2)n^2 - (r-4)n}{2}$ where $n \in \mathbb{Z}^+$. [8]

2. [Maximum marks: 28]

This question asks you to explore cubic polynomials of the form $(x-r)(x^2-2ax+a^2+b^2)$ for $x \in \mathbb{R}$ and corresponding cubic equations with one real root and two complex roots of the form $(z-r)(z^2-2az+a^2+b^2)=0$ for $z \in \mathbb{C}$.

In parts (a), (b) and (c), let $r=1$, $a=4$ and $b=1$.

Consider the equation $(z-1)(z^2-8z+17)=0$ for $z \in \mathbb{C}$.

(a) (i) Given that 1 and $4+\mathrm{i}$ are roots of the equation, write down the third root. [1]

(ii) Verify that the mean of the two complex roots is 4. [1]

Consider the function $f(x)=(x-1)(x^2-8x+17)$ for $x \in \mathbb{R}$.

(b) Show that the line $y=x-1$ is tangent to the curve $y=f(x)$ at the point $\mathrm{A}(4, 3)$. [4]

(c) Sketch the curve $y=f(x)$ and the tangent to the curve at point A, clearly showing where the tangent crosses the x-axis. [2]

Consider the function $g(x)=(x-r)(x^2-2ax+a^2+b^2)$ for $x \in \mathbb{R}$ where $r, a \in \mathbb{R}$ and $b \in \mathbb{R}, b>0$.

(d) (i) Show that $g'(x)=2(x-r)(x-a)+x^2-2ax+a^2+b^2$. [2]

(ii) Hence, or otherwise, prove that the tangent to the curve $y=g(x)$ at the point $\mathrm{A}(a, g(a))$ intersects the x-axis at the point $\mathrm{R}(r, 0)$. [6]

The equation $(z-r)(z^2-2az+a^2+b^2)=0$ for $z \in \mathbb{C}$ has roots r and $a \pm b\mathrm{i}$ where $r, a \in \mathbb{R}$ and $b \in \mathbb{R}, b>0$.

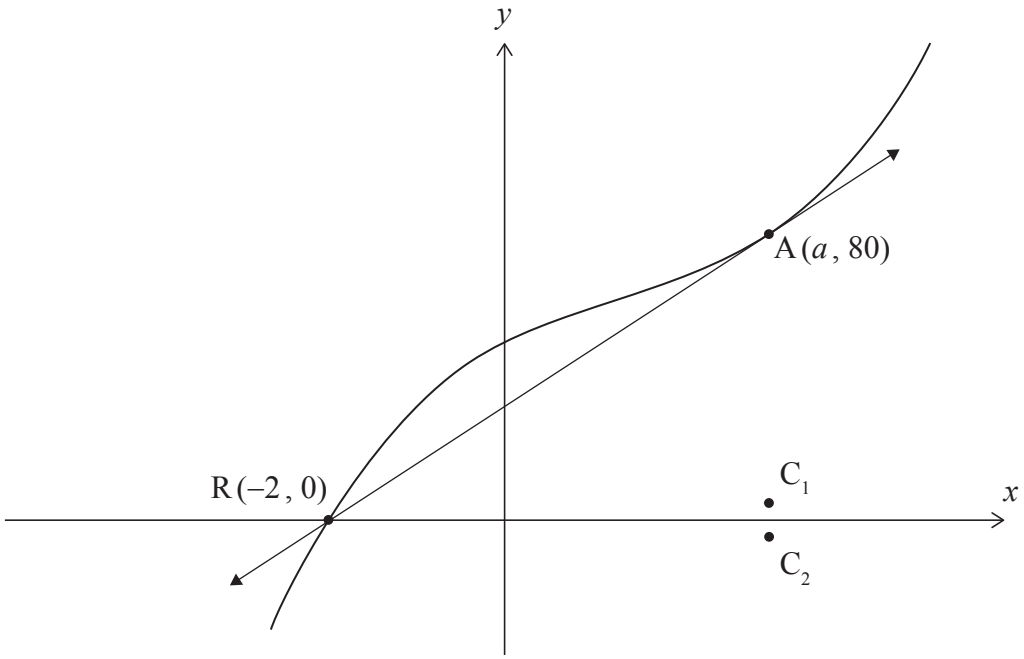
(e) Deduce from part (d)(i) that the complex roots of the equation $(z-r)(z^2-2az+a^2+b^2)=0$ can be expressed as $a \pm \mathrm{i}\sqrt{g'(a)}$. [1]

(This question continues on the following page)

(Question 2 continued)

On the Cartesian plane, the points $C_1\left(a, \sqrt{g'(a)}\right)$ and $C_2\left(a, -\sqrt{g'(a)}\right)$ represent the real and imaginary parts of the complex roots of the equation $(z-r)(z^2-2az+a^2+b^2)=0$.

The following diagram shows a particular curve of the form $y=(x-r)(x^2-2ax+a^2+16)$ and the tangent to the curve at the point $A(a, 80)$. The curve and the tangent both intersect the x-axis at the point $R(-2, 0)$. The points C_1 and C_2 are also shown.



(f) (i) Use this diagram to determine the roots of the corresponding equation of the form $(z-r)(z^2-2az+a^2+16)=0$ for $z \in \mathbb{C}$. [4]

(ii) State the coordinates of C_2. [1]

Consider the curve $y=(x-r)(x^2-2ax+a^2+b^2)$ for $a \neq r$, $b>0$. The points $A\left(a, g(a)\right)$ and $R(r, 0)$ are as defined in part (d)(ii). The curve has a point of inflexion at point P.

(g) (i) Show that the x-coordinate of P is $\frac{1}{3}(2a+r)$.

You are **not** required to demonstrate a change in concavity. [2]

(ii) Hence describe numerically the horizontal position of point P relative to the horizontal positions of the points R and A. [1]

Consider the special case where $a=r$ and $b>0$.

(h) (i) Sketch the curve $y=(x-r)(x^2-2ax+a^2+b^2)$ for $a=r=1$ and $b=2$. [2]

(ii) For $a=r$ and $b>0$, state in terms of r, the coordinates of points P and A. [1]

References:

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